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A primer of Perron–Frobenius theory for matrix polynomials

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Abstract

We present an extension of Perron–Frobenius theory to the spectra and numerical ranges of Perron polynomials, namely, matrix polynomials of the form

$$L(\lambda) = I\lambda^m - A_{m-1}\lambda^{m-1} - \cdots - A_1\lambda - A_0,$$

where the coefficient matrices are entrywise nonnegative. Our approach relies on the companion matrix linearization. First, we recount the generalization of the Perron–Frobenius Theorem to Perron polynomials and report some of its consequences. Subsequently, we examine the role of $L(\lambda)$ in multistep difference equations and provide a multistep version of the Fundamental Theorem of Demography. Finally, we extend Issos' results on the numerical range of nonnegative matrices to Perron polynomials.

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1. Introduction

Matrices with nonnegative entries arise in a wide variety of areas including dynamical systems theory, economics, statistics and optimization to name a few. The basic

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tenant of their theory, that the spectral radius of a nonnegative matrix is necessarily an eigenvalue, is historically attributed to Perron and Frobenius. Hence the term Perron–Frobenius theory is frequently being used to describe the development of a spectral theory for nonnegative matrices and, more generally, for matrices that leave a proper cone invariant. The effort to extend the reach of nonnegative matrices has led to various generalizations and to a systematic study of related matrix classes. Here we explore the extension of Perron–Frobenius theory to (monic) matrix polynomials of the form

$$L(\lambda) = I\lambda^m - A_{m-1}\lambda^{m-1} - \cdots - A_1\lambda - A_0. \tag{1.1}$$

When the matrices A_j $(j = 0, 1, \ldots, m-1)$ in (1.1) are entrywise nonnegative $n \times n$ matrices, we shall refer to $L(\lambda)$ as an $n \times n$ *Perron polynomial* of degree m. Recall that the spectrum of a matrix polynomial $L(\lambda)$ consists of the scalars λ such that $\det L(\lambda) = 0$ and so it coincides with the spectrum of A_0 when $m = 1$. It is therefore natural to seek a generalization of Perron–Frobenius theory to Perron polynomials.

Entrywise nonnegativity associated with generalized eigenproblems has been studied before; see e.g., [2,21] where matrix pencils are considered. An extension of Perron–Frobenius theory to positive operators on Banach lattices is pursued in [29]. The peripheral spectrum of the monic polynomial in (1.1) when the coefficients are positive (compact) operators in a Banach lattice are considered in [7,18,27]. In addition, the spectral properties of Perron polynomials are examined in [8,9] via a partial linearization based on expansion graphs that were introduced in [10]. Finally, elements of a spectral theory for Perron polynomials appear in the study of the stability of linear difference equations with positive coefficients [24].

Our present goal is to take a comprehensive look at Perron–Frobenius theory for Perron polynomials, based on the companion matrix global linearization. The presentation is organized as follows: In Section 2 we recall definitions, notation and include some preliminaries regarding matrix polynomials. Section 3 contains a recount of Perron–Frobenius theory for Perron polynomials. Specifically, it contains the extension of the two classical parts of the Perron–Frobenius Theorem as well as a discussion of irreducibility, primitivity, stochastic Perron polynomials and bounds for the spectral radius. We also discuss $L(\lambda)^{-1}$ and consider its positivity, relating our discussion to the theory of M-matrices. In Section 4 we consider the association of $L(\lambda)$ in (1.1) with the multistep matrix difference equations

$$u_{j+m} = A_{m-1}u_{j+m-1} + \cdots + A_1u_{j+1} + A_0u_j \quad (j = 0, 1, \ldots),$$

which is of interest in the study of higher order linear differential equations, dynamical systems theory, economic models and queuing theory [4,13,24,28]. We provide a generalization of the Fundamental Theorem of Demography (see [17]) pursuant to the asymptotic behavior of u_j. Finally, in Section 5, we extend the results on the numerical range of nonnegative matrices to Perron polynomials; Section 5 concludes with two illustrative examples and some directions for future study.

2. Definitions and preliminaries

Given a vector $z = [z_i] \in \mathbb{C}^n$, we use the norms $\|z\|_1 = \sum_{i=1}^n |z_i|$ and $\|z\|_2 = \sqrt{\sum_{i=1}^n |z_i|^2}$. By $\mathbf{1}$ we denote an all ones vector of appropriate size. The spectrum of $A \in \mathbb{C}^{n\times n}$ is denoted by $\sigma(A)$ and its *spectral radius* by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

The *nonnegative orthant* in $\mathbb{R}^n$, that is, the set of all nonnegative vectors in $\mathbb{R}^n$, is denoted by $\mathbb{R}^n_+$. Entrywise ordering of arrays of the same size is indicated by $\geqslant$. We write $A > B$ if A, B are real and every entry of $A - B$ is positive. When $A \geqslant 0$ (resp., $A > 0$), we refer to A as *nonnegative* (resp., *positive*). An *M-matrix* is a square matrix of the form $M = sI - A$, where $A \geqslant 0$ and $s > \rho(A)$. It is well known [3, Chapter 6] that M-matrices are inverse nonnegative, that is, $M^{-1} \geqslant 0$.

We call a square matrix A *irreducible* if there does not exist a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square, nonvacuous matrices. We call A *k-cyclic* (or just *cyclic*) if for some permutation matrix P,

$$PAP^{\mathrm{T}} = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k,1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.1}$$

where the zero blocks along the diagonal are square. Notice that a k-cyclic matrix is also m-cyclic for any divisor m of k. The largest positive integer k for which A is k-cyclic is referred to as the *cyclic index* of A.

The *directed graph*, $\mathscr{D}(A)$, of $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ consists of the set of vertices $\{1, 2, \ldots, n\}$ and a set of directed edges (i, j) connecting vertex i to vertex j if and only if $a_{ij} \neq 0$. We say vertex i has *access* to vertex j if there is a path $(i, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, j)$ connecting the two vertices. We say $\mathscr{D}(A)$ is *strongly connected* if any two distinct vertices i, j have access to each other. It is well known that A is irreducible if and only if $\mathscr{D}(A)$ is strongly connected. A *cycle of length k* in $\mathscr{D}(A)$ consists of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$, where the vertices $i_1, i_2, \ldots, i_k$ are distinct. The nonzero diagonal entries of A correspond to cycles of length 1 in $\mathscr{D}(A)$.

The main clause of the Perron–Frobenius Theorem [1,3,14,31] states that if $A \geqslant 0$, then $\rho(A) \in \sigma(A)$. An irreducible matrix $A \geqslant 0$ is called *primitive* if $\rho(A)$ is the only eigenvalue of A of maximum modulus. Note that the following are equivalent statements for a nonnegative irreducible matrix A: (i) A is primitive; (ii) the cyclic index of A equals 1; (iii) the greatest common divisor of the cycle lengths in $\mathscr{D}(A)$ equals 1; (iv) $A^m > 0$ for some positive integer m.

We continue with some preliminary facts on matrix polynomials. A scalar λ_0 is said to be an *eigenvalue* of $L(\lambda)$ in (1.1) if the system $L(\lambda_0)y = 0$ has a nonzero solution $y_0 \in \mathbb{C}^n$; y_0 is referred to as a *right eigenvector* of $L(\lambda)$ corresponding to λ_0. A nonzero vector w_0 is said to be a *left eigenvector* of $L(\lambda)$ corresponding to λ_0 if $L(\lambda_0)^{\mathrm{T}} w_0 = 0$. The *spectrum* of $L(\lambda)$ is the set of all eigenvalues of $L(\lambda)$, $\sigma(L) = \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\}$. The *companion matrix* of $L(\lambda)$,

$$C_L = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{m-1} \end{bmatrix} \in \mathbb{C}^{nm \times nm}, \tag{2.2}$$

provides a *linearization* of $L(\lambda)$ (see [11,16]); this means that

$$E(\lambda)(I\lambda - C_L)G(\lambda) = \begin{bmatrix} L(\lambda) & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}, \tag{2.3}$$

where $E(\lambda)$ and $G(\lambda)$ are $nm \times nm$ matrix polynomials with constant nonzero determinants.

More specifically,

$$E(\lambda) = \begin{bmatrix} E_1(\lambda) & E_2(\lambda) & \cdots & E_{m-1}(\lambda) & I \\ -I & 0 & \cdots & 0 & 0 \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \end{bmatrix}$$

with

$$\begin{aligned} E_{m-1}(\lambda) &= I\lambda - A_{m-1} \\ E_{m-2}(\lambda) &= I\lambda^2 - A_{m-1}\lambda - A_{m-2} \\ &\vdots \\ E_2(\lambda) &= I\lambda^{m-2} - A_{m-1}\lambda^{m-3} - \cdots - A_3\lambda - A_2 \\ E_1(\lambda) &= I\lambda^{m-1} - A_{m-1}\lambda^{m-2} - \cdots - A_2\lambda - A_1, \end{aligned} \tag{2.4}$$

and

$$G(\lambda) = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ I\lambda & I & \cdots & 0 & 0 \\ I\lambda^2 & I\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I\lambda^{m-1} & I\lambda^{m-2} & \cdots & I\lambda & I \end{bmatrix}.$$

As a consequence of (2.3) and the nature of $E(\lambda)$ and $G(\lambda)$, the spectrum of $L(\lambda)$ and its linearization coincide, that is, $\sigma(L) = \sigma(C_L)$. The *spectral radius* of $L(\lambda)$ is $\rho(L) = \max\{|\lambda| : \lambda \in \sigma(L)\}$, and thus it also coincides with the spectral radius of C_L, $\rho(C_L)$. An eigenvalue $\lambda \in \sigma(L) = \sigma(C_L)$ is referred to as *maximal* if $|\lambda| = \rho(L)$.

3. Perron–Frobenius for polynomials

The relation between the eigenvectors of $L(\lambda)$ in (1.1) and the eigenvectors of the companion matrix C_L in (2.2) is obtained directly via (2.3) as follows.

Lemma 3.1. *Let μ be an eigenvalue of the matrix polynomial $L(\lambda)$ in* (1.1) *and C_L be its companion matrix. Then*

(i) *a nonzero vector $y \in \mathbb{C}^n$ is a right eigenvector of $L(\lambda)$ corresponding to $\mu \in \sigma(L)$ if and only if $\begin{bmatrix} y \\ \mu y \\ \vdots \\ \mu^{m-1} y \end{bmatrix} \in \mathbb{C}^{nm}$ is a right eigenvector of C_L corresponding to μ;*

(ii) *a nonzero vector $w \in \mathbb{C}^n$ is a left eigenvector of $L(\lambda)$ corresponding to $\mu \in \sigma(L)$ if and only if $\begin{bmatrix} E_1(\mu)^{\mathrm{T}} w \\ E_2(\mu)^{\mathrm{T}} w \\ \vdots \\ E_{m-1}(\mu)^{\mathrm{T}} w \\ w \end{bmatrix} \in \mathbb{C}^{nm}$ is a left eigenvector of C_L corresponding to μ; the matrix polynomials $E_1(\lambda), E_2(\lambda), \ldots, E_{m-1}(\lambda)$ are those defined in* (2.4).

The Perron–Frobenius Theorem for nonnegative matrices can now be stated for matrix polynomials.

Theorem 3.2 (Perron–Frobenius, Part I). *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *Then the following hold*:

(a) $\rho(L) \in \sigma(L)$.
(b) *$L(\lambda)$ has an entrywise nonnegative right eigenvector and an entrywise nonnegative left eigenvector correspondingto $\rho(L)$.*
(c) *$\rho(L)$ is a nondecreasing function of the entries of each A_j* ($j = 0, 1, \ldots, m-1$).

If, in addition, the companion matrix C_L is irreducible, then the following hold:

(a′) $\rho(L)$ *is an algebraically simple eigenvalue of* $L(\lambda)$.
(b′) $L(\lambda)$ *has an entrywise positive right eigenvector and an entrywise positive left eigenvector corresponding to* $\rho(L)$.
(c′) $\rho(L)$ *is an increasing function of the entries of each* A_j $(j = 0, 1, \ldots, m-1)$.

Proof. Recall that $\sigma(L) = \sigma(C_L)$. Statements (a) and (b) follow directly from Lemma 3.1 and [14, Theorem 8.3.1], and statement (c) from [14, Theorem 8.1.18]. Lemma 3.1 and [14, Theorem 8.4.4] yield immediately (a′) and (b′). Statement (c′) follows from the first part of [14, Theorem 8.4.5]. □

Theorem 3.3 (Perron–Frobenius, Part II). *Let* $L(\lambda)$ *be an* $n \times n$ *Perron polynomial as in* (1.1) *and suppose its companion matrix* C_L *is irreducible. Then there exists an integer* $1 \leqslant k \leqslant nm$ *such that the maximal eigenvalues of* $L(\lambda)$ *are the roots of* $\lambda^k - \rho(L)^k = 0$. *Moreover,*

$$\sigma(L) = \mathrm{e}^{\mathrm{i}(2t\pi/k)}\sigma(L) \quad (t = 0, 1, \ldots, k-1)$$

and C_L *is cyclic of index* k. *When* $k = 1$, C_L *is a primitive matrix.*

Proof. The result follows directly from [3, Chapter 2, Theorem 2.20] applied to C_L. □

Next we consider the irreducibility of the companion matrix, raised as a condition in Theorems 3.2 and 3.3.

Theorem 3.4. *The companion matrix* C_L *in* (2.2) *of an mth degree* $n \times n$ *matrix polynomial* $L(\lambda)$ *as in* (1.1) *is irreducible if and only if each one of the vertices* $\{n(m-1)+1, \ldots, nm\}$ *in* $\mathscr{D}(C_L)$ *has access to each one of the vertices* $\{1, 2, \ldots, n\}$. *In particular, if* A_0 *is irreducible, so is* C_L.

Proof. Group the vertices in $\mathscr{D}(C_L)$ in m sets

$$J_k = \{kn+1, kn+2, \ldots, kn+n\} \quad (k = 0, 1, \ldots, m-1).$$

If C_L is irreducible, clearly every vertex in J_{m-1} has access to every vertex in J_0. To prove the converse, assume that every vertex in J_{m-1} has access to every vertex in J_0. Since every vertex in J_k for $k = 1, 2, \ldots, m-1$ is accessed by some vertex in J_0, it follows that every vertex in J_{m-1} has access to every vertex in $\mathscr{D}(C_L)$. In addition, each of the vertices in J_k for $k = 0, 1, \ldots, m-2$ has access to every vertex in J_{m-1}. Hence, $\mathscr{D}(C_L)$ is strongly connected and C_L is irreducible. The condition for irreducibility of C_L clearly holds when A_0 is irreducible. □

Remark 3.5. The specific block structure of an irreducible companion matrix C_L does not seem to be of any tractable consequence to its primitivity. C_L is primitive if and only if the greatest common divisor of the cycle lengths in $\mathscr{D}(C_L)$ is 1, e.g., if trace$(A_{m-1}) > 0$ and C_L is irreducible, then C_L is primitive.

We proceed with a generalization of the notion of a stochastic matrix. Recall that a (row) stochastic matrix is a nonnegative square matrix A that satisfies $A\mathbf{1} = \mathbf{1}$. Let then $L(\lambda)$ be the $n \times n$ matrix polynomial in (1.1). We call $L(\lambda)$ *stochastic* if $A_j \geqslant 0$ $(j = 0, 1, \ldots, m-1)$ and $L(1)\mathbf{1} = 0$; that is, $L(\lambda)$ is a Perron polynomial having $1 \in \sigma(L)$ with corresponding eigenvector $\mathbf{1}$. Note that $L(\lambda)$ is stochastic if and only if C_L is a stochastic matrix. As a consequence, we have the following extension of a well-known fact about nonnegative matrices.

Theorem 3.6. *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *Suppose that $\rho := \rho(L) > 0$ and that $L(\rho)z = 0$ for some vector $z > 0$. Then*

$$\hat{L}(\lambda) = I\lambda^m - \frac{A_{m-1}}{\rho}\lambda^{m-1} - \cdots - \frac{A_1}{\rho^{m-1}}\lambda - \frac{A_0}{\rho^m}$$

is similar to a stochastic matrix polynomial.

Proof. Let $L(\lambda), \rho, z = [z_i]$ be as prescribed and let $D = \operatorname{diag}\{z_1, z_2, \ldots, z_n\}$. Let also

$$D_L = \begin{bmatrix} D & 0 & \cdots & 0 \\ 0 & \rho D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho^{m-1} D \end{bmatrix} \in \mathbb{R}^{nm \times nm}.$$

Notice that by Lemma 3.1, $D_L^{-1} C_L D_L \mathbf{1} = \rho\mathbf{1}$; i.e., $D_L^{-1} C_L D_L / \rho$ is a stochastic matrix. It follows that the matrix polynomial $D^{-1}\hat{L}(\lambda)D$ is stochastic. $\square$

Next we review some results and bounds for the spectral radius of a Perron polynomial relative to the rational matrix function

$$S(\lambda) = A_{m-1} + \frac{1}{\lambda}A_{m-2} + \cdots + \frac{1}{\lambda^{m-2}}A_1 + \frac{1}{\lambda^{m-1}}A_0. \tag{3.1}$$

In the next proposition the operators $\min(\cdot)$ and $\max(\cdot)$ applied to a vector return the minimum and maximum entry of that vector, respectively.

Proposition 3.7. *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *Then*

$$\min\{\min(S(1)\mathbf{1}), 1\} \leqslant \rho(L) \leqslant \max\{\max(S(1)\mathbf{1}), 1\}. \tag{3.2}$$

Moreover, if the companion matrix of $L(\lambda)$ is irreducible, equality in any one of the inequalities in (3.2) *holds if and only if $L(\lambda)$ is a stochastic matrix polynomial.*

Proof. The two inequalities and the equality cases follow from [3, Chapter 2, Theorem 2.35] applied to C_L. $\square$

The following proposition obtains directly from [27, Proposition 2.1, Corollary 2.2] and [7, Proposition 1.1]. The main idea used in the proofs of the cited results is indeed the monotonicity of $\rho(S(\mu))$ as a function of $\mu \in (0, \infty)$ (a nonincreasing function).

Proposition 3.8. *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *Let $S(\lambda)$ be as in* (3.1) *and denote $\rho := \rho(L)$. Then the following hold:*

(i) *The function $\phi : (0, \infty) \longrightarrow \mathbb{R}$ defined by $\phi(\mu) = \rho(S(\mu))$ is continuous and nonincreasing. Moreover, $\rho = 0$ if and only if $\phi \equiv 0$; if $\rho > 0$, then $\rho = \phi(\rho)$.*
(ii) *$\rho(S(1)) \leqslant \rho \leqslant \rho(S(1))^{1/m}$ if $\rho(S(1)) \leqslant 1$.*
(iii) *$\rho(S(1))^{1/m} \leqslant \rho \leqslant \rho(S(1))$ if $\rho(S(1)) \geqslant 1$.*
(iv) *$\rho < 1$ if and only if $\rho(S(1)) < 1$.*
(v) *$\rho = 1$ if and only if $\rho(S(1)) = 1$.*
(vi) *$\rho > 0$ if and only if there exists a $\tau > 0$ such that $\tau = \rho(S(\tau))$; in this case $\tau = \rho$.*

Next we extend a basic component of Perron–Frobenius theory. A Perron polynomial of degree $m = 1$ is of the form $L(\lambda) = I\lambda - A$, where $A \geqslant 0$. Thus, for $\lambda > \rho(A) = \rho(L)$, $L(\lambda)$ is an M-matrix and $L(\lambda)^{-1} = (I\lambda - A)^{-1} \geqslant 0$. An analogous result holds for Perron polynomials of arbitrary degree as shown below (see also [7, Proposition 2.1]).

Theorem 3.9. *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *For every $\mu > \rho(L)$, $L(\mu)$ is a nonsingular M-matrix and*

$$L(\mu)^{-1} = \sum_{k=0}^{\infty} \frac{PC_L^k R}{\mu^{k+1}} \geqslant 0,$$

where

$$P = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{n\times nm} \quad \textit{and} \quad R = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \in \mathbb{C}^{nm\times n}.$$

Proof. First, from [12, Proposition 5.1.2], for every $\mu \notin \sigma(L)$, the inverse of $L(\mu)$ and the resolvent of its companion matrix C_L satisfy $L(\mu)^{-1} = P(I\mu - C_L)^{-1}R$. Now observe that for $\mu > \rho(L) = \rho(C_L)$, $I\mu - C_L$ is an M-matrix and so $(I\mu - C_L)^{-1} \geqslant 0$. It follows that for $\mu > \rho(L)$, $L(\mu)$ has nonpositive off-diagonal entries and is inverse positive. That is, $L(\mu)$ is a nonsingular M-matrix; see [3, Chapter 6, Theorem 2.3]. In particular, using the Neumann expansion for $(I\mu - C_L)^{-1}$ when $\mu > \rho(L) = \rho(C_L)$, we obtain

$$L(\mu)^{-1} = P(I\mu - C_L)^{-1}R = \mu^{-1}P\left(I - \frac{C_L}{\mu}\right)^{-1} R = \sum_{k=0}^{\infty} \frac{PC_L^k R}{\mu^{k+1}}. \qquad \square$$

4. Multistep matrix difference equations

Motivated by iterative methods and the ubiquitous nature of the difference equations

$$u_{j+1} = Au_j \quad (j = 0, 1, \ldots),$$

where A is a nonnegative matrix (e.g., in population dynamics), the convergence of the powers of a nonnegative matrix have been studied extensively. A classical result is the following.

Theorem 4.1. *Let $A \geqslant 0$ be a primitive matrix with spectral radius $\rho := \rho(A)$. Let $y, w > 0$ be right and left eigenvectors of A corresponding to ρ, respectively, such that $w^{\mathrm{T}}y = 1$. Then*

(i)
$$\lim_{j\to\infty} \frac{A^j}{\rho^j} = yw^{\mathrm{T}}.$$

(ii) *Referring to the process $u_{j+1} = Au_j$ $(j = 0, 1, \ldots)$, $u_0 \neq 0$, we have that*

$$\lim_{j\to\infty} \|u_j\|_1 = \begin{cases} 0 & \textit{if } \rho < 1 \\ \|(w^{\mathrm{T}}u_0)y\|_1 & \textit{if } \rho = 1. \\ \infty & \textit{if } \rho > 1 \end{cases}$$

A proof of Theorem 4.1(i) can be found in [3,14,17]. Part (ii) is referred to as the Fundamental Theorem of Demography (see [5, Theorem 1.1.2] and [17]).

As mentioned in the introduction, the generalization from matrices to matrix polynomials $L(\lambda)$ as in (1.1) is relevant to a matrix model of the form

$$u_{j+m} = A_{m-1}u_{j+m-1} + \cdots + A_1u_{j+1} + A_0u_j \quad (j = 0, 1, \ldots), \tag{4.1}$$

where $u_0, u_1, \ldots, u_{m-1} \in \mathbb{R}^n$ are initial states that determine the solution $\{u_j\}_{j=0}^{\infty}$. In this section we will establish a generalization of Theorem 4.1 to the multistep difference equations in (4.1). Indeed, if C_L is the companion matrix of $L(\lambda)$, then (4.1) is equivalent to

$$x_{j+1} = C_Lx_j \quad (j = 0, 1, \ldots),$$

where

$$x_j = \begin{bmatrix} u_j \\ u_{j+1} \\ \vdots \\ u_{j+m-1} \end{bmatrix} \quad (j = 0, 1, \ldots).$$

By [11, Theorem 1.6], for a given initial vector x_0, (4.1) has the unique solution

$$u_j = \underbrace{[I \quad 0 \quad \cdots \quad 0]}_{\in \mathbb{C}^{n\times nm}} C_L^j x_0 \quad (j = 0, 1, \ldots). \tag{4.2}$$

Recall that $A \in \mathbb{R}^{n\times n}$ is nonnegative if and only if $A\mathbb{R}^n_+ \subseteq \mathbb{R}^n_+$. Similarly, one can easily establish that for the scheme in (4.1) and the proper cone

$$W^{n,m}_+ = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \cdots \times \mathbb{R}^n_+$$

the $n \times n$ matrix polynomial $L(\lambda)$ in (1.1) is Perron if and only if $x_j \geqslant 0$ ($j = 0, 1, \ldots$) whenever $x_0 \geqslant 0$. Our analysis yields the following interesting generalization of the Fundamental Theorem of Demography.

Theorem 4.2. *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1)*, and let C_L be primitive. Suppose $y > 0$ and $w > 0$ are right and left eigenvectors of $L(\lambda)$ corresponding to $\rho := \rho(L)$, respectively, normalized so that*

$$[w^{\mathrm{T}} E_1(\rho) \quad \cdots \quad w^{\mathrm{T}} E_{m-1}(\rho) \quad w^{\mathrm{T}}] \begin{bmatrix} y \\ \rho y \\ \vdots \\ \rho^{m-1} y \end{bmatrix} = 1,$$

where $E_1(\lambda), \ldots, E_{m-1}(\lambda)$ are defined in (2.4)*. Consider a nonzero initial vector* $x_0 = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \in \mathbb{C}^{nm}$ *and let $\{u_0, u_1, \ldots\}$ be the solution of* (4.1) *given in* (4.2)*. Taking $E_m(\lambda) = I$, we have*

$$\lim_{j\to\infty} \frac{1}{\rho^j} u_j = \left[w^{\mathrm{T}} \left(\sum_{k=1}^{m} E_k(\rho) u_{k-1} \right) \right] y. \tag{4.3}$$

Furthermore,

$$\lim_{j\to\infty} \|u_j\|_1 = \begin{cases} 0 & \text{if } \rho < 1 \\ \left\| \left[w^{\mathrm{T}} \left(\sum_{k=1}^{m} E_k(\rho) u_{k-1} \right) \right] y \right\|_1 & \text{if } \rho = 1 \\ \infty & \text{if } \rho > 1 \end{cases}.$$

Proof. Since the nonnegative companion matrix C_L is primitive, by Lemma 3.1 and by [14, Theorem 8.5.1],

$$\begin{aligned}
&\lim_{j\to\infty}\left[\frac{1}{\rho(C_L)}C_L\right]^j \\
&= \begin{bmatrix} y \\ \rho y \\ \vdots \\ \rho^{m-1} y\end{bmatrix}[w^{\mathrm{T}}E_1(\rho) \quad \cdots \quad w^{\mathrm{T}}E_{m-1}(\rho) \quad w^{\mathrm{T}}] \\
&= \begin{bmatrix} yw^{\mathrm{T}}E_1(\rho) & \cdots & yw^{\mathrm{T}}E_{m-1}(\rho) & yw^{\mathrm{T}} \\ \rho yw^{\mathrm{T}}E_1(\rho) & \cdots & \rho yw^{\mathrm{T}}E_{m-1}(\rho) & \rho yw^{\mathrm{T}} \\ \vdots & & \vdots & \vdots \\ \rho^{m-1}yw^{\mathrm{T}}E_1(\rho) & \cdots & \rho^{m-1}yw^{\mathrm{T}}E_{m-1}(\rho) & \rho^{m-1}yw^{\mathrm{T}}\end{bmatrix}.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\lim_{j\to\infty}\frac{1}{\rho^j}u_j \\
&= \underbrace{[I \quad 0 \quad \cdots \quad 0]}_{\in\mathbb{C}^{n\times nm}}\lim_{j\to\infty}\left[\frac{1}{\rho^j}C_L^j\right]x_0 \\
&= [I \quad 0 \quad \cdots \quad 0] \\
&\quad\times\begin{bmatrix} yw^{\mathrm{T}}E_1(\rho) & \cdots & yw^{\mathrm{T}}E_{m-1}(\rho) & yw^{\mathrm{T}} \\ \rho yw^{\mathrm{T}}E_1(\rho) & \cdots & \rho yw^{\mathrm{T}}E_{m-1}(\rho) & \rho yw^{\mathrm{T}} \\ \vdots & & \vdots & \vdots \\ \rho^{m-1}yw^{\mathrm{T}}E_1(\rho) & \cdots & \rho^{m-1}yw^{\mathrm{T}}E_{m-1}(\rho) & \rho^{m-1}yw^{\mathrm{T}}\end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1}\end{bmatrix} \\
&= \left[yw^{\mathrm{T}}E_1(\rho) \quad \cdots \quad yw^{\mathrm{T}}E_{m-1}(\rho) \quad yw^{\mathrm{T}}\right]\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1}\end{bmatrix} \\
&= y\left(w^{\mathrm{T}}E_1(\rho)u_0 + \cdots + w^{\mathrm{T}}E_{m-1}(\rho)u_{m-2} + w^{\mathrm{T}}u_{m-1}\right),
\end{aligned}$$

completing the proof. □

5. The numerical range of a Perron polynomial

In two recent papers [19,23], the results in James Nestor Issos' thesis [20] on the numerical range of an entrywise nonnegative matrix were reviewed, extended and utilized. In this section we generalize these results to the numerical range of a Perron polynomial. For that purpose, consider an $n \times n$ polynomial $L(\lambda)$ of degree m as in (1.1) and the corresponding $nm \times nm$ companion matrix C_L in (2.2). The *numerical range* of $L(\lambda)$ is defined by

$$W(L) = \{\lambda \in \mathbb{C} : x^* L(\lambda)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\},$$

which is a compact subset of $\mathbb{C}$ that contains the spectrum $\sigma(L)$. For a linear pencil $I\lambda - A$, where $A \in \mathbb{C}^{n\times n}$, $W(I\lambda - A)$ coincides with the classical *numerical range* (or *field of values*) of the matrix A, namely, $F(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$, and it is always convex (see e.g., [15]). The *numerical radius* of $L(\lambda)$ is defined by $r(L) = \max\{|\lambda| : \lambda \in W(L)\}$. In the remainder of this paper, we assume that $r(L) > 0$, or equivalently, that at least one of the coefficients A_j $(j = 0, 1, \ldots, m-1)$ of $L(\lambda)$ is nonzero.

Denote the *symmetric part* of a real square matrix A by $H(A) = (A + A^{\mathrm{T}})/2$. Also, for any $\mu \in \mathbb{C}$ and $x \in \mathbb{C}^n$, denote

$$y(\mu, x) = \begin{bmatrix} x \\ \mu x \\ \vdots \\ \mu^{m-1}x \end{bmatrix} \in \mathbb{C}^{nm}$$

and observe that $\|y(\mu, x)\|_2 = \|x\|_2\sqrt{1 + |\mu| + \cdots + |\mu|^{m-1}}$. Vectors of the form $y(\mu, x)$ have been used to prove that the numerical range of the companion matrix C_L always contains $W(L)$ and the origin [22, Proposition 2.4]. Based on the same approach, we obtain in the next two lemmas the relation between the numerical range and the companion matrix of an arbitrary monic matrix polynomial $L(\lambda)$, which is essential for the further investigation of $W(L)$.

Lemma 5.1. *The numerical range of* $L(\lambda)$ *in* (1.1) *satisfies*

$$W(L)\backslash\{0\} = \left\{ \mu \neq 0 : \mu = y(\mu, x)^* C_L y(\mu, x),\ x \in \mathbb{C}^n, \right.$$
$$\left. \|x\|_2 = \frac{1}{\sqrt{1 + |\mu| + \cdots + |\mu|^{m-1}}} \right\}.$$

Proof. For any scalar $\mu \in \mathbb{C}$ and for any nonzero vector $x \in \mathbb{C}^n$, by the results in [22], we have

$$\begin{aligned} &y(\mu, x)^*(I\mu - C_L)y(\mu, x) \\ &\quad = \begin{bmatrix} x^* & \overline{\mu}x^* & \cdots & \overline{\mu}^{m-1}x^* \end{bmatrix} \end{aligned}$$

$$\times \begin{bmatrix} I\mu & -I & 0 & \cdots & 0 \\ 0 & I\mu & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -I \\ -A_0 & -A_1 & -A_2 & \cdots & I\mu - A_{m-1} \end{bmatrix} \begin{bmatrix} x \\ \mu x \\ \vdots \\ \mu^{m-1}x \end{bmatrix}$$
$$= \overline{\mu}^{m-1} x^* L(\mu) x.$$

Thus, a nonzero $\mu \in W(L)$ if and only if $\mu = y(\mu, x)^* C_L y(\mu, x)$ for some nonzero $x \in \mathbb{C}^n$ with norm $\|x\|_2 = (1 + |\mu| + \cdots + |\mu|^{m-1})^{-1/2}$. □

Lemma 5.2. *If the matrix coefficients of $L(\lambda)$ in* (1.1) *are real and $\mu_0 = \max\{\mu : \mu \in W(L) \cap \mathbb{R}\}$, then $F(L(\mu_0))$ lies in the closed right half-plane of $\mathbb{C}$ and the symmetric part $H(L(\mu_0))$ is singular positive semidefinite.*

Proof. By the definitions of the numerical range of a matrix and of a matrix polynomial, it is clear that $0 \in F(L(\mu_0))$, and that for every $\mu > \mu_0$, $0 \notin F(L(\mu))$. Moreover, the numerical range of any real matrix $L(\mu)$ ($\mu \geqslant \mu_0$) is convex and symmetric with respect to the real axis, and depends continuously on μ with respect to the Hausdorff metric. Consequently, $F(L(\mu_0))$ lies in the right closed half-plane of $\mathbb{C}$ and has the origin as a boundary point. Hence, the numerical range of the symmetric part of $L(\mu_0)$, namely,

$$F(H(L(\mu_0))) = \{\operatorname{Re}\lambda : \lambda \in F(L(\mu_0))\}$$

is a real interval with the origin as its left endpoint. Thus, $H(L(\mu_0))$ is singular positive semidefinite and the proof is complete. □

Next we generalize Theorem 3.2 in [23].

Theorem 5.3. *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1). *Then $r(L) \in W(L)$ and there exists a nonnegative vector $x_r \in \mathbb{R}^n$ such that*

$$\|x_r\|_2 = \frac{1}{\sqrt{1 + r(L) + \cdots + r(L)^{m-1}}} \quad \textit{and} \quad x_r^{\mathrm{T}} L(r(L)) x_r = 0.$$

If, in addition, 0 *is a simple eigenvalue of $H(L(r(L)))$, then a vector $\hat{x} \in \mathbb{C}^n$ satisfies*

$$\|\hat{x}\|_2 = \frac{1}{\sqrt{1 + r(L) + \cdots + r(L)^{m-1}}} \quad \textit{and} \quad \hat{x}^* L(r(L)) \hat{x} = 0$$

if and only if $\hat{x} = \mathrm{e}^{\mathrm{i}\theta} x_r$ for some $\theta \in [0, 2\pi)$.

Proof. Let $\mu \in W(L)$. Then for every $x \in \mathbb{C}^n$ such that $\|x\|_2 = (1 + |\mu| + \cdots + |\mu|^{m-1})^{-1/2}$ and $\mu = y(\mu, x)^* C_L y(\mu, x)$, we readily have that

$$
\begin{aligned}
|\mu| &= |y(\mu, x)^* C_L y(\mu, x)| \\
&\leqslant |y(\mu, x)|^{\mathrm{T}} C_L |y(\mu, x)| \\
&= \begin{bmatrix} |x|^{\mathrm{T}} & |\mu||x|^{\mathrm{T}} & \cdots & |\mu|^{m-1}|x|^{\mathrm{T}} \end{bmatrix} C_L \begin{bmatrix} |x| \\ |\mu||x| \\ \vdots \\ |\mu|^{m-1}|x| \end{bmatrix}.
\end{aligned}
$$

Since the vectors x and $|x|$ have the same norm, by Lemma 5.1, it follows that the nonnegative number

$$
\begin{bmatrix} |x|^{\mathrm{T}} & |\mu||x|^{\mathrm{T}} & \cdots & |\mu|^{m-1}|x|^{\mathrm{T}} \end{bmatrix} C_L \begin{bmatrix} |x| \\ |\mu||x| \\ \vdots \\ |\mu|^{m-1}|x| \end{bmatrix} \geqslant |\mu|
$$

also lies in $W(L)$. Hence, the numerical radius $r(L)$ belongs to $W(L)$ and there is a nonnegative vector $x_r \in \mathbb{R}^n$ such that $\|x_r\|_2 = (1 + r(L) + \cdots + r(L)^{m-1})^{-1/2}$ and $x_r^{\mathrm{T}} L(r(L)) x_r = 0$.

By Lemma 5.2, 0 is the smallest eigenvalue of $H(L(r(L)))$, and thus, by [15, Lemma 1.5.7], a nonzero vector $\hat{x} \in \mathbb{C}^n$ satisfies $\hat{x}^* L(r(L))\hat{x} = 0$ if and only if it is an eigenvector of $H(L(r(L)))$ corresponding to 0. Hence, if 0 is a simple eigenvalue of $H(L(r(L)))$, then $\hat{x}$ is a scalar multiple of x_r. $\square$

The following two results are direct generalizations of Lemmas 3.3 and 3.4 in [23], respectively.

Corollary 5.4. *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1). *Suppose* 0 *is a simple eigenvalue of $H(L(r(L)))$ and $x_r \geqslant 0$ is the vector in Theorem* 5.3, *and let $\mathrm{e}^{\mathrm{i}\theta} r(L) \in W(L)$ for some $\theta \in (0, 2\pi)$. Then for every nonzero vector $x_\theta \in \mathbb{C}^n$ such that*

$$
\|x_\theta\|_2 = \frac{1}{\sqrt{1 + r(L) + \cdots + r(L)^{m-1}}} \quad \textit{and} \quad x_\theta^* L(\mathrm{e}^{\mathrm{i}\theta} r(L)) x_\theta = 0,
$$

we have that $|x_\theta| = x_r$.

Proof. Let $x_\theta \in \mathbb{C}^n$ be any nonzero vector such that $x_\theta^* L(e^{\mathrm{i}\theta} r(L)) x_\theta = 0$. Then as in the proof of Theorem 5.3,

$$
\begin{aligned}
r(L) = |\mathrm{e}^{\mathrm{i}\theta} r(L)| &= \left| y(\mathrm{e}^{\mathrm{i}\theta} r(L), x_\theta)^* C_L y(\mathrm{e}^{\mathrm{i}\theta} r(L), x_\theta) \right| \\
&\leqslant \begin{bmatrix} |x_\theta|^{\mathrm{T}} & r(L)|x_\theta|^{\mathrm{T}} & \cdots & r(L)^{m-1}|x_\theta|^{\mathrm{T}} \end{bmatrix} C_L \begin{bmatrix} |x_\theta| \\ r(L)|x_\theta| \\ \vdots \\ r(L)^{m-1}|x_\theta| \end{bmatrix} \\
&\leqslant r(L).
\end{aligned}
$$

Consequently, the vector $|x_\theta|$ satisfies

$$\||x_\theta|\|_2 = \frac{1}{\sqrt{1 + r(L) + \cdots + r(L)^{m-1}}} \quad \text{and} \quad |x_\theta|^{\mathrm{T}} L(r(L))|x_\theta| = 0,$$

and by Theorem 5.3, $|x_\theta| = x_r$. $\square$

Theorem 5.5. *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1). *Suppose that* 0 *is a simple eigenvalue of $H(L(r(L)))$ and $x_r \geqslant 0$ is the vector in Theorem* 5.3. *If there are angles $\theta, \phi \in [0, 2\pi)$ such that $\mathrm{e}^{\mathrm{i}\theta} r(L), \mathrm{e}^{\mathrm{i}\phi} r(L) \in W(L)$, then $\mathrm{e}^{\mathrm{i}(\theta+\phi)} r(L)$ also lies in $W(L)$.*

Proof. By Corollary 5.4, there exist two nonzero vectors $x_\theta, x_\phi \in \mathbb{C}^n$ such that

$$|x_\theta| = |x_\phi| = x_r \quad \text{and} \quad x_\theta^* L(\mathrm{e}^{\mathrm{i}\theta} r(L)) x_\theta = x_\phi^* L(\mathrm{e}^{\mathrm{i}\phi} r(L)) x_\phi = 0.$$

By the relations

$$\begin{aligned} r(L) = |\mathrm{e}^{\mathrm{i}\theta} r(L)| &= |y(\mathrm{e}^{\mathrm{i}\theta} r(L), x_\theta)^* C_L y(\mathrm{e}^{\mathrm{i}\theta} r(L), x_\theta)| \\ &= y(r(L), |x_\theta|)^{\mathrm{T}} C_L y(r(L), |x_\theta|) \end{aligned}$$

and

$$\begin{aligned} r(L) = |\mathrm{e}^{\mathrm{i}\phi} r(L)| &= |y(\mathrm{e}^{\mathrm{i}\phi} r(L), x_\phi)^* C_L y(\mathrm{e}^{\mathrm{i}\phi} r(L), x_\phi)| \\ &= y(r(L), |x_\phi|)^{\mathrm{T}} C_L y(r(L), |x_\phi|) \end{aligned}$$

and following exactly the steps in the proof of [23, Lemma 3.4], one can construct a vector $w \in \mathbb{C}^n$ such that $|w| = x_r$ and

$$\mathrm{e}^{\mathrm{i}(\theta+\phi)} r(L) = y(\mathrm{e}^{\mathrm{i}(\theta+\phi)} r(L), w)^* C_L y(\mathrm{e}^{\mathrm{i}(\theta+\phi)} r(L), w)$$

or equivalently,

$$w^* L(\mathrm{e}^{\mathrm{i}(\theta+\phi)} r(L)) w = 0. \qquad \square$$

Suppose the maximal elements of $W(L)$ are of the form

$$r(L)\mathrm{e}^{\mathrm{i}\theta_j} \quad (j = 1, 2, \ldots, k)$$

with $\theta_1 = 0$ and $\theta_2, \ldots, \theta_k \in (0, 2\pi)$ for some positive integer k. Then, by Theorem 5.5, the set

$$\{\theta_1 (\operatorname{mod} 2\pi), \theta_2 (\operatorname{mod} 2\pi), \ldots, \theta_k (\operatorname{mod} 2\pi)\}$$

is a (finite) additive abelian group and hence a cyclic group. As a consequence, we have the following generalization of [23, Theorem 3.5].

Corollary 5.6. *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1). *Suppose that* 0 *is a simple eigenvalue of $H(L(r(L)))$ and that $W(L)$ has k maximal elements. Then these maximal elements are of the form*

$$r(L)\mathrm{e}^{\mathrm{i}(2t\pi/k)} \quad (t = 0, 1, \ldots, k-1).$$

We proceed with an illustration of the numerical range of a Perron polynomial. Let A be an $n \times n$ nonnegative matrix with irreducible symmetric part $H(A)$, and consider the Perron polynomial of degree m,

$$L_A(\lambda) = I\lambda^m - A.$$

Suppose A is k-cyclic. Then by the Perron–Frobenius Theorem and the results in [23],

$$\mathrm{e}^{\mathrm{i}(2t\pi/k)}\sigma(A) = \sigma(A) \quad \text{with } \mathrm{e}^{\mathrm{i}(2t\pi/k)}\rho(A) \in \sigma(A) \quad (t = 0, 1, \ldots, k-1)$$

and

$$\mathrm{e}^{\mathrm{i}(2t\pi/k)}F(A) = F(A) \quad \text{with } \mathrm{e}^{\mathrm{i}(2t\pi/k)}r(A) \in F(A) \quad (t = 0, 1, \ldots, k-1).$$

The spectrum and the numerical range of $L_A(\lambda)$ are given, respectively, by

$$\sigma(L_A) = \{\mu \in \mathbb{C} : \mu^m \in \sigma(A)\} \quad \text{and} \quad W(L_A) = \{\mu \in \mathbb{C} : \mu^m \in F(A)\}.$$

Since the mth roots of a nonzero complex number $\alpha\mathrm{e}^{\mathrm{i}\theta}$ $(\alpha > 0,\ \theta \in [0, 2\pi))$ are $\alpha^{1/m}\mathrm{e}^{\mathrm{i}((2\xi\pi+\theta)/m)}$ $(\xi = 0, 1, \ldots, m-1)$ and the mth roots of $\mathrm{e}^{\mathrm{i}(2t\pi/k)}$ are given by $\mathrm{e}^{\mathrm{i}((2\pi(t+\xi m))/km)}$ $(\xi = 0, 1, \ldots, m-1)$, we have that

$$\mathrm{e}^{\mathrm{i}(2\zeta\pi/km)}\sigma(L_A) = \sigma(L_A) \quad \text{with } \mathrm{e}^{\mathrm{i}(2\zeta\pi/km)}\rho(L_A) \in \sigma(L_A)$$
$$(\zeta = 0, 1, \ldots, km-1)$$

and

$$\mathrm{e}^{\mathrm{i}(2\zeta\pi/km)}W(L_A) = W(L_A) \quad \text{with } \mathrm{e}^{\mathrm{i}(2\zeta\pi/km)}r(L_A) \in W(L_A)$$
$$(\zeta = 0, 1, \ldots, km-1).$$

Example 1. The nonnegative matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is 2-cyclic and has an irreducible symmetric part. The boundary of the (convex) numerical range of A is drawn in the left part of Fig. 1. The numerical range of the Perron polynomial $L_A(\lambda) = I\lambda^3 - A$ is illustrated in the right part of the figure (the unshaded star-shaped region). It is obtained using the inclusion–exclusion algorithm described in [25] and confirms the above discussion. The eigenvalues of A and $L_A(\lambda)$ are marked with +'s.

Example 2. For the matrix A in the previous example, consider the Perron polynomial $L(\lambda) = I\lambda^3 - I\lambda^2 - I\lambda - A$. The numerical range $W(L)$ is illustrated in

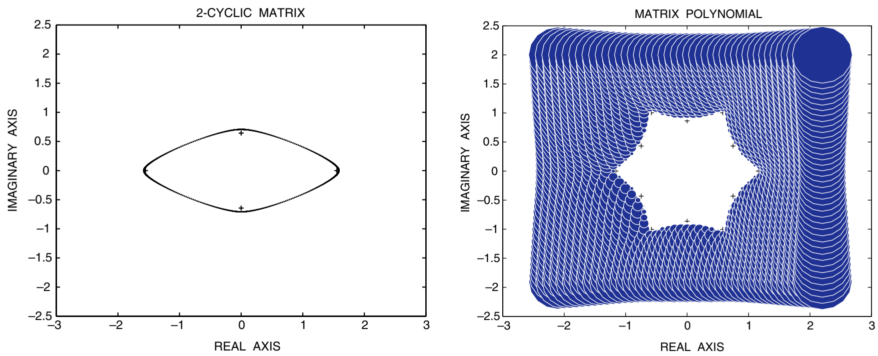

Fig. 1. Spectra and numerical ranges of A and $I\lambda^3 - A$.

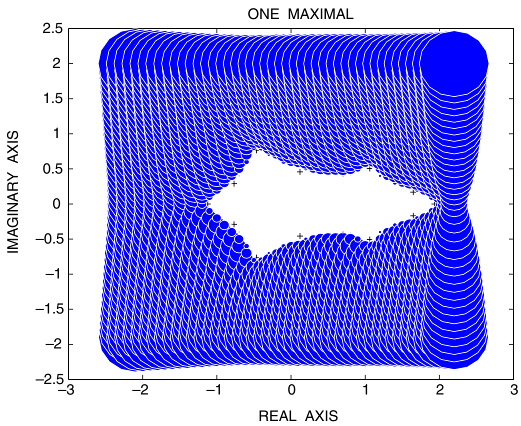

Fig. 2. Spectrum and numerical range of $L(\lambda)$.

Fig. 2 and has exactly one (real positive) maximal element. The eigenvalues of $L(\lambda)$ are marked with +'s. Note that the (nonnegative) companion matrix of $L(\lambda)$,

$$C_L = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ A & I & I \end{bmatrix}$$

is primitive and the spectrum $\sigma(L) = \sigma(C_L)$ has also exactly one maximal element, which is approximately 1.9331.

In conclusion, we have pursued a discussion on the spectrum and the numerical range of a Perron polynomial, as well as an asymptotic analysis of the associated multistep finite differences scheme. Our discussion is based on the companion matrix linearization. We have not treated a number of subjects in Perron–Frobenius theory suitable for further research. They include a study of the Perron generalized eigenspace in the spirit of the results of Rothblum [26] (see also [30]) on the existence of a nonnegative basis with combinatorial structure. For such a direction it may be advantageous to consider the role of expansion graphs as presented in [8,9], where the level and height characteristics are examined. It is also worthwhile pursuing a partition of Perron polynomials analogous to the partition of Z-matrices and Z-pencils in classes based on the spectral radii of submatrices, see [6,21], respectively.

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